Technical Notes

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A Parametric Study of Falkner-Skan Problem with Mass Transfer

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IN this Note, a parametric differentiation method is used to study a particular Falkner-Skan problem,[1,2] flow over an infinite wedge of angle $\beta\pi$, with mass transfer taking place in the form of either suction or injection at the boundaries. When the wedge angle is zero ($\beta = 0$), it is the Blasius flat-plate flow; and when the wedge angle increases to $\pi(\beta = 1)$, it is the stagnation flow with the wedge surfaces at right angles to the flow. Solutions of the boundary-layer type flow with fluid acceleration ($\beta > 0$) or deceleration ($\beta < 0$) and with mass transfer at the wedge surfaces have been given mostly numerically using multiple-step predictor-corrector type integration formulas[3,4] involving, essentially, trial and error procedures in transforming the boundary-value problem to one of initial-value amenable to computations. The trial and error procedure may be avoided, however, if the parametric method is applied.

The parametric method has been applied successfully to several boundary layer problems,[5] including the Falkner-Skan problem.[6] Basically, it transforms the original nonlinear boundary-layer equation into a linear one and calculates, through a sequence of small perturbations, the rate of change of the solution to the original problem with respect to a suitable parameter. The parameter may appear in the differential equation, the boundary conditions, or both. In the present study, the pressure-gradient parameter β or the mass-transfer parameter K, to be defined later, may be chosen as the parameter. It then enables one to study the continuous change in the flow variable due either to the change in wedge angle or in the rate of mass transfer at the surfaces.

Of the three governing ordinary differential equations for momentum, diffusion, and energy which depict the Falkner-Skan problem with mass transfer,[3] the momentum equation is a nonlinear third-order equation, whereas the other two are linear and second order and their solutions are incumbent upon that of the momentum equation. Only the solution to the nonlinear momentum equation is intended herein.

Using the usual nomenclature as in Refs. 2 and 4, the momentum equation may be written as

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

where f is the nondimensional stream function of η, defined as

$$\eta = y[U(x)/(2-\beta)\nu x]^{1/2}$$

Here, x and y are distances along and normal to the wedge surface, respectively; $U(x)$, the freestream velocity; and ν, the kinematic viscosity.

The pertinent bounding conditions are as follows:

$$\eta = 0: f = -K, f' = 0 \qquad \eta \to \infty: f' \to 1 \tag{2}$$

where K is the mass transfer parameter given by

$$K = v_w[(2-\beta)x/\nu U(x)]^{1/2}$$

and v_w is the rate of mass transfer at the wedge surfaces. Here, for $\beta < 2$, $K > 0$ ($v_w > 0$) indicates blowing and $K < 0$ ($v_w < 0$) denotes suction at the surfaces.

Equation (1) is a nonlinear equation of $f(\eta)$ which depends implicitly on two parameters β and K; the latter appears in the boundary conditions (2). Hence, $f(\eta)$ may also be written as $f(\eta, \beta, K)$. Since we can vary only one parameter at a time in using the parametric method, we may choose K as the varying parameter while keeping β constant, or vice versa.

Differentiating both the governing Eq. (1) and the boundary conditions (2) with respect to K while keeping β constant and defining

$$F^{(n)}(\eta, \beta, K) = \partial f^{(n)}(\eta, \beta, K)/\partial K \tag{3}$$

where n denotes the order of derivatives with respect to η, Eqs. (1) and (2) become

$$\begin{gathered} F''' + fF'' + f''F - 2\beta f'F' = 0 \\ F(0) = -1, \ F'(0) = 0, \ F'(\infty) = 0 \end{gathered} \tag{4}$$

Equation (4) is a linear equation in F with f and its derivatives appearing as varying coefficients. It is observed that the nonlinearity of the problem is now confined only to the first-order ordinary Eq. (3). The resulting problem may still be difficult to solve but nevertheless considerably more tractable than the original nonlinear problem, especially in view of the fact that approximate methods for linear differential equations are better developed than those for nonlinear equations.

A numerical integration of Eq. (4) is started at $\beta = K = 0$, corresponding to the Blasius flat-plate flow problem, where the function $f(\eta)$ and its derivative may be obtained by the Piercy-Preston successive iteration scheme[7] starting with an assumed f profile to give

$$f'(\eta) = \int_0^\eta \exp\left(-\int_0^\zeta f d\zeta\right) d\zeta \bigg/ \int_0^\infty \exp\left(-\int_0^\zeta f d\zeta\right) d\zeta \tag{5}$$

and

$$f(\zeta) = \int_0^\eta f'(\zeta) d\zeta \tag{6}$$

With the coefficients known, Eq. (4) may then be integrated, for example, with a simple second-order Runge-Kutta scheme to obtain $F^{(n)}(\eta, 0, 0)$. Equation (3) may then be numerically integrated with a small ΔK interval to yield $f^{(n)}(\eta, 0, \Delta K)$. With the known coefficients $f^{(n)}(\eta, 0, \Delta K)$, Eq. (4) can again be integrated to give $F^{(n)}(\eta, 0, \Delta K)$ which, in turn, yields $f^{(n)}(\eta, 0, 2\Delta K)$ by numerically integrating Eq. (3). The procedure may be repeated until a desired K value is reached.

The aforementioned scheme is limited to cases of $\beta = 0$ and $K \leqq 0.877$, where the upper limit of K is that critical value corresponding to $\beta = 0$ at which separation at the surfaces occurs. It should be remarked that since $f^{(n)}(\eta, \beta, 0)$ are not readily available, integration of Eq. (4) for nonzero β values via the parametric method cannot be performed for lack of a starting solution.

The integration of Eq. (4), being a boundary-value problem, may be facilitated by first transforming it into an initial-value problem. In view of the linearity of Eq. (4) in F, we can hypothetically set up two problems, F_1 and F_2, similar to Eq. (4) as

$$\begin{gathered} F_j''' + fF_j'' + f''F_j - 2\beta f'F_j = 0 \qquad (j = 1 \text{ or } 2) \\ F_j(0) = -1, \ F_j'(0) = 0, \ \begin{cases} F_1''(0) = 0 \\ F_2''(0) = 1 \end{cases} \end{gathered} \tag{7}$$

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Both being initial-value problems they can be readily solved numerically. Since, in general, $F_1''(0) = 0$ and $F_2''(0) = 1$ are not exactly equal to $F''(0)$ for which $F'(\infty) \to 0$, Eq. (7) yields non-vanishing values of $F_1'(\infty)$ and $F_2'(\infty)$. It can then be shown through superposition that $F''(0)$ is related to $F_1'(\infty)$ and $F_2'(\infty)$ by

$$F''(0) = F_1'(\infty)/[F_1'(\infty) - F_2'(\infty)] \tag{8}$$

Thus by solving the two fictitious problems, Eqs. (7), and having determined $F''(0)$ according to Eq. (8), the solution to the problem itself, Eq. (4), becomes "exact" insofar as no trial and error procedures are involved.

A parametric study of the problem with respect to β, while keeping K constant can also be made in a similar manner. For such cases, however, since $f(0) = -K$, Eq. (6) appears with an additional term as

$$f(\eta) = \int_0^\eta f'(\zeta)\,d\zeta - K \tag{9}$$

This scheme, however, is limited to $K \leqq 0.877$ in that no starting solutions of $f(\eta, 0, K)$ for $K > 0.877$ are physically available.

The above parametric studies with β and K could be used to cover the ranges of $\beta \geqq 0$ and $-\infty < K \leqq 0.877$. For cases with K greater than 0.877 but less than the critical value corresponding to a particular nonzero value of $\beta > 0$, say β_0, however, they could be solved in two steps. Starting at $\beta = K = 0$ a parametric study with β could be initiated to obtain $f^{(n)}(\eta, \beta_0, 0)$. Having found $f^{(n)}(\eta, \beta_0, 0)$, and hence the coefficients for Eq. (4) with $\beta = \beta_0$, the parametric study could be continued with K as the varying parameter until $f^{(n)}(\eta, \beta_0, K)$, with K below the critical value corresponding to β_0, are obtained.

Results and Discussion

In this study, the ranges of the parameters considered are $0 \leqq \beta \leqq 1.0$ and $-4 \leqq K \leqq 0.87$. For the parametric study with K, while keeping $\beta = 0$, the outer boundary was set at $\eta_\infty = 20$ with $\Delta\eta = 0.025$ for the range $0 \leqq K \leqq 0.87$, and at $\eta_\infty = 10$ with $\Delta\eta = 0.005$ for $-4 \leqq K \leqq 0$. Integration with respect to K was carried out from 0 to -4 at $\Delta K = -0.01$; while integration in the positive range of K was performed at different step sizes: $\Delta K = 0.01$ for $0 \leqq K \leqq 0.6$, $K = 0.002$ for $0.6 \leqq K \leqq 0.8$, $K = 0.001$ for $0.8 \leqq K \leqq 0.85$, and $\Delta K = 0.00025$ for $0.85 \leqq K \leqq 0.87$. The calculated values of $f''(0, 0, K)$ are shown in Table 1 together with the values obtained by Elzy and Sisson[4] using the multiple-step predictor-corrector type numerical integration formulas. Despite the use of a simple second-order Runge-Kutta integration scheme, the present parametric study gives results of $f''(0, 0, K)$ which agree to at least three decimal places with those calculated by Elzy and Sisson. It should be remarked that for $K < 0$, while a fairly large $\Delta K = -0.01$ was satisfactory, a very small $\Delta\eta$ had to be employed. In fact, it was found that a further decrease in the step size of K does not substantially improve the accuracy of the results, and in order to achieve high accuracy the step size in η had to be successively reduced as K becomes more negative. For $K > 0$, however, because of the thickened boundary-layer thickness with increasing values of K, a larger step size of $\Delta\eta = 0.025$ was found to be satisfactory while the step size in K became critical and had to be successively reduced as K approaches the critical values of 0.877.

The parametric study with β while keeping K constant was carried out at $K = -2, -1, 0, 0.4$, and 0.8 for $0 \leqq \beta \leqq 1$. The step sizes used were $\Delta\eta = 0.005$ at $\eta_\infty = 10$ for $K \geqq 0$, $\Delta\eta = 0.0025$ at $\eta_\infty = 5$ for $K < 0$, and $\Delta\beta = 0.01$ in all cases. The calculated values of $f''(0, \beta, K)$ are shown in Table 2 together with those of Elzy and Sisson.[4] Again, a three-decimal-place accuracy can be claimed, except at higher values of $K_0 > 0$ where the outer boundary of $\eta_\infty = 10$ is apparently not large enough and the step size of $\Delta\beta = 0.01$ is not sufficiently small. For negative and small positive values of K_0, it was found that $\Delta\beta = 0.01$ was adequate and that further decreases in $\Delta\beta$ do not substantially improve the accuracy of the results.

Table 1 Values of $f''(0, 0, K)$[a]

K	Present method	Elzy & Sisson[b]	K	Present method	Elzy & Sisson[b]
−4.0	4.1143	4.1146	0.2	0.3305	0.3305
−3.5	3.6280	...	0.3	0.2658	0.2657
−3.0	3.1449	3.1451	0.4	0.2049	0.2048
−2.5	2.6664	...	0.5	0.1485	0.1484
−2.0	2.1944	2.1945	0.6	0.0975	0.0974
−1.5	1.7318	...	0.7	0.0531	...
−1.0	1.2836	1.2836	0.8	0.0175	0.0175
−0.5	0.8579	...	0.85	0.0045	...
0	0.4696[c]	0.4696	0.87	0.0008	...
0.1	0.3986	0.3985			

[a] K is used as the parameter of differentiation.
[b] Ref. 4.
[c] Starting solution, obtained by Piercy-Preston iteration method.

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Table 2 Values of $f''(0, \beta, K)$[a]

β	$K = -2$		$K = -1$		$K = 0$		$K = 0.4$		$K = 0.8$	
	Present method	Elzy & Sisson[b]	Present method	Elzy & Sisson[b]	Present method	Elzy & Sisson[b]	Present method	Elzy & Sisson[b]	Present method	Elzy & Sisson[b]
0	2.1945[c]	2.1945	1.2836[c]	1.2836	0.4696[c]	0.4696	0.2048[c]	0.2048	0.0174[c]	0.0175
0.1	2.2497	2.2497	1.3611	1.3610	0.5874	0.5870	0.3462	0.3455	0.1762	0.1749
0.2	2.3028	2.3027	1.4331	1.4329	0.6873	0.6867	0.4561	0.4552	0.2852	0.2840
0.3	2.3540	2.3539	1.5006	1.5003	0.7754	0.7747	0.5498	0.5488	0.3770	0.3759
0.4	2.4034	2.4032	1.5642	1.5638	0.8552	0.8544	0.6330	0.6320	0.4583	0.4573
0.5	2.4512	2.4510	1.6246	1.6241	0.9285	0.9276	0.7087	0.7077	0.5323	0.5313
0.6	2.4975	2.4973	1.6821	1.6816	0.9966	0.9958	0.7785	0.7775	0.6006	0.5997
0.7	2.5425	2.5422	1.7371	1.7366	1.0606	1.0598	0.8438	0.8428	0.6645	0.6636
0.8	2.5862	2.5859	1.7899	1.7894	1.1211	1.1202	0.9052	0.9042	0.7246	0.7238
0.9	2.6288	2.6285	1.8407	1.8402	1.1785	1.1777	0.9634	0.9624	0.7817	0.7809
1.0	2.6703	2.6700	1.8898	1.8893	1.2334	1.2325	1.0188	1.0179	0.8362	0.8354

[a] β is used as the parameter of differentiation.
[b] Ref. 4.
[c] Starting solution, obtained by Piercy-Preston iteration method.

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Exact Solution for Dynamic Oscillations of Re-Entry Bodies

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Nomenclature

A	= reference area of re-entry body
a	= parameter in confluent hypergeometric function, Eq. (9)
C_D, C_L, C_m	= standard aerodynamic drag, lift and moment coefficients
$C_{L_\alpha} = \dfrac{\partial C_L}{\partial \alpha}$	= aerodynamic lift coefficient slope vs angle of attack
$C_{m_\alpha} = \dfrac{\partial C_m}{\partial \alpha}$	= aerodynamic moment coefficient slope vs angle of attack
$C_{m_q} = \dfrac{\partial C_m}{\partial (qL/v)}$	= damping moment ($C_{m_q} < 0$) due to pitching angular velocity
C_{N_α}	= $C_D + C_{L_\alpha}$
${}_1F_1$	= confluent hypergeometric function, Eq. (10)
I	= mass moment of inertia in pitch
J_0	= zero order Bessel function of the first kind
k_1, k_2, k_3	= Allen's aerodynamic stability parameters, Eqs. (3, 4, and 5)
L	= characteristic reference length of re-entry body
m	= mass of re-entry body
q	= angular velocity in pitch relative to Earth
v	= velocity along trajectory of re-entry body
y	= altitude from reference level where $\rho = \rho_0$ when $y = 0$, Eq. (1)
$Y = \beta y$	= nondimensional altitude
$Z = Z_0 e^{-Y}$	= variable replacing nondimensional altitude, Eq. (8)
α	= angle of attack of re-entry body relative to its trajectory
β	= atmospheric density exponential factor, Eq. (1)
δ_0	= nondimensional mass ratio, Eq. (6)
$\theta_E = -\gamma_0 > 0$	= downward angle between straight line re-entry flight path and Earth's surface
ρ	= atmospheric air density

FRIEDRICH and Dore[1] and Allen[2] independertly obtained a Bessel's function solution that approximated the dynamic oscillations of a hypersonic re-entry body that is descending through a planetary atmosphere. In order to establish the range of validity of this Bessel function approximation we have derived an exact solution of Allen's differential equation for a straight line trajectory. This exact solution is a special case of the confluent hypergeometric function that allows the appropriate criteria for dynamic stability to be formulated explicitly in terms of the basic aerodynamic coefficients. The Bessel function approximation is shown to provide an excellent solution for the linearized oscillations of the usual ballistic missile. The exact solution is found to be necessary only when one considers re-entry bodies which have such a small aerodynamic restoring moment that the special case of near critical damping is approached. This condition of near critical damping is studied, and the relations between the aerodynamic coefficients that will achieve this condition are derived.

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Allen's Differential Equation and its Exact Solution

Allen[2] derived the linearized differential equation for the angle-of-attack oscillations of a rotationally symmetric re-entry body that had a sufficiently large drag so that the descending flight path was essentially a straight line. He also assumed that the aerodynamic force and moment coefficients remained constant, so his analysis would be most applicable to the hypersonic case. The altitude range to be considered was such that the acceleration due to gravity could be considered to be a constant independent of altitude, and the variation of the Earth's atmospheric density could be approximated by the exponential function

$$\rho/\rho_0 = e^{-\beta y} = e^{-Y}, \quad \beta^{-1} \approx 22{,}000 \text{ ft} \approx 6710 \text{ m} \tag{1}$$

Under these assumptions Allen obtained the following differential equation, where the angle of attack variation (α) may be considered to be in either pitch, yaw or any vector sum of these two angular displacements

$$d^2\alpha/dY^2 + 2k_1 e^{-Y} d\alpha/dY + (k_2 e^{-Y} + k_3 e^{-2Y})\alpha = 0 \tag{2}$$

$$k_1 = (\delta_0/2)[C_D - C_{L_\alpha} + (C_{m_q} + C_{m_{\dot\alpha}})(mL^2/I)] \tag{3}$$

$$k_2 = \delta_0[C_{L_\alpha} - C_{m_\alpha}(mL^2/I)(\beta L \sin\theta_E)^{-1}] \tag{4}$$

$$k_3 = \delta_0^2 C_{L_\alpha}[-C_D - C_{m_q}(mL^2/I)] \tag{5}$$

These are in Allen's[2] notation except that we have replaced the constant Allen designated as k_0 by

$$\delta_0 = \rho_0 A (2\beta m \sin\theta_E)^{-1} = k_0/2C_D \tag{6}$$

We have found that the exact solution of Eq. (2) could be written as

$$\alpha(Y) = \alpha_0\, {}_1F_1(a, 1, Z) \exp[(k_1/Z_0) - (\tfrac{1}{2})]Z \tag{7}$$

$$Z(Y) = 2(k_1^2 - k_3)^{1/2} e^{-Y} = Z_0 e^{-Y} \tag{8}$$

$$a = (\tfrac{1}{2}) - (k_1 + k_2)Z_0^{-1} \tag{9}$$

where

$${}_1F_1(a, 1, Z) = 1 + aZ + a(a+1)(2!)^{-2}Z^2 + a(a+1)(a+2)(3!)^{-2}Z^3 + \dots \tag{10}$$

is a special case of the confluent hypergeometric function, e.g. see Slater.[3] This function is an infinite series except when a is a negative integer so that $a = -n$, and in this case ${}_1F_1$ reduces to a finite polynomial of order n. For values of $-a > 10$ we can replace this confluent hypergeometric function by its asymptotic form when $a \to -\infty$ in terms of the zero-order Bessel function (J_0) of the first kind as follows:

$${}_1F_1(a, 1, Z) \approx e^{Z/2} J_0\{[(2-4a)Z]^{1/2}\} \tag{11}$$

It will be shown that the parameter a defines the number of oscillations that will occur. For example, when $a = -n$ is a negative integer, then the exact solution given by Eq. (7) cannot have more than n oscillations. Consequently, small negative values of a yield a motion that resembles the motion produced by near critical damping in oscillating systems. However, the usual ballistic missile has $-a > 10$ and in these cases Eq. (11) provides an excellent approximation that reduces Eq. (7) to

$$\alpha(Y) = \alpha_0 J_0\{[(2-4a)Z]^{1/2}\} \exp(k_1 Z/Z_0) \tag{12}$$

This is identical to Allen's[2] approximate solution to Eq. (2), and it